B A D A N I A O P E R A C Y J N E I D E C Y Z J E

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Barbara GŁADYSZ*
Dorota KUCHTA*

MINIMISATION OF THE EXPECTED NUMBER OF LATE JOBS IN A SINGLE MACHINE SYSTEM WITH FUZZY PROCESSING TIMES AND FUZZY DUE DATES

One machine scheduling problem with fuzzy processing times and fuzzy due dates is considered. A simple method for determining a fuzzy set of the number of late jobs is presented. The optimal sequence is defined as one minimizing the expected number of late jobs. Special case is discussed in which the problem can be reduced to a well-known and easy to solve optimization problem.

1. Introduction

We consider a one-machine scheduling problem with fuzzy processing times. This problem has already been studied in the literature [2], [3], [5], but the optimal processing sequence can be determined using various criteria. Here, we will be interested in minimising the total number of late jobs, but as this number will be fuzzy, we choose the expected value of the fuzzy number of late jobs as optimisation criterion.

The fuzziness of the processing times represents incomplete information – the processing times will be known exactly only in the future, but the evaluation of the problem, especially of the risk of not finishing the processing of all the jobs on time, has to be carried out when full information is not available yet. Such situations are common in practice, thus there is a great need for models and algorithms for scheduling problems in case of incomplete or imprecise information. This paper is a contribution to this field.

We will use standard definitions of fuzzy number and fuzzy operations on fuzzy numbers. The fuzziness will be denoted by means of sign ~.

* Institute of Industrial Engineering and Management, Wrocław University of Technology, ul. Smoluchowskiego 25, 50-372 Wrocław, dorota.kuchta@pwr.wroc.pl, barbara.gladysz@pwr.wroc.pl

2. Formulation of the problem

Let us consider a one-machine system in which n jobs $Z_1, Z_2, \ldots, Z_n$ should be processed.

For each job Z_s the following information is given:

- $\tilde{T}_s$ – fuzzy normalized job processing time being a fuzzy number with the membership function $\mu_{\tilde{T}_s}(x)$,
- $\tilde{D}_s$ – fuzzy normalized job due date being a fuzzy number with the membership function $\mu_{\tilde{D}_s}(x)$, including the special (crisp) case when $\mu_{\tilde{D}_s}(x)$ is equal to 1 for exactly one x and to 0 otherwise.

We assume that all the jobs are ready to be processed at moment 0.

Our aim is to determine such a processing schedule $p^* = (Z_i, Z_{i_2}, \ldots, Z_{i_n})$ for which the expected value of the fuzzy set of the total number of late jobs will be minimal (expected value for fuzzy sets is defined in [1]).

Permutation p will be defined by means of 0–1 decision variables $x_{sk}(p)$ $(s = 1, ..., n; k = 1, ..., n)$ which fulfil the following condition:

$$\sum_{s=1}^{n} x_{sk}(p) = 1 \quad \text{and} \quad \sum_{k=1}^{n} x_{sk}(p) = 1$$

and $x_{sk}(p)$ takes on value 1 when job Z_s is processing in the k-th position for permutation p and value 0 otherwise.

3. Expected number of late jobs

Let us denote by $\tilde{Y}_{Z_s(k)}(p)$ the moment when, for a given permutation $p = (Z_{i_1}, Z_{i_2}, \ldots, Z_{i_{k-1}}, Z_s, Z_{i_{k+1}}, \ldots, Z_{i,_n})$, the processing time of job Z_s in the k-th position is finished.

$$\tilde{Y}_{Z_s(k)}(p) = \tilde{T}_{i_1} \tilde{+} \tilde{T}_{i_2} \tilde{+} \ldots \tilde{+} \tilde{T}_{i_{k-1}} \tilde{+} \tilde{T}_s .$$

$\tilde{Y}_{Z_s(k)}(p)$ is a fuzzy number with the membership function $\mu_{\tilde{Y}_{Z_s(k)}(p)}(x)$.

Let us define the function

$$L(y,d)=\begin{cases}1 \text{ for } y>d\\ 0 \text{ for } y\le d\end{cases} \tag{1}$$

If we consider, for a fixed *k*, all $y\in\tilde{Y}_{Z_s(k)}(p)$ and all $d\in\tilde{D}_s$, function (1) generates for a given permutation *p*, according to the Zadeh extension principle, a fuzzy set of the following form:

$$\tilde{L}_{z_s(k)}(p)=0/\mu_{0_{Z_s(k)}}(p)+1/\mu_{1_{Z_s(k)}}(p) \tag{2}$$

where:

$$\mu_{0_{Z_s(k)}}(p)=\sup_{L(y,d)=0}\min\left(\mu_{\tilde{Y}_{Z_s(k)}(p)}(y),\mu_{\tilde{D}_s}(d)\right)$$

$$\mu_{1_{Z_s(k)}}(p)=\sup_{L(y,d)=1}\min\left(\mu_{\tilde{Y}_{Z_s(k)}(p)}(y),\mu_{\tilde{D}_s}(d)\right)$$

The fuzzy set (2) takes on value 1 if job Z_s is late and value 0 if it is not. It is a normalized fuzzy set

$$\text{if}\quad \mu_{0_{Z_s(k)}}(p)<1\quad\text{then}\quad \mu_{1_{Z_s(k)}}(p)=1$$

and the other way round:

$$\text{if}\quad \mu_{1_{Z_s(k)}}(p)<1\quad\text{then}\quad \mu_{0_{Z_s(k)}}(p)=1\,.$$

Let us define the following function:

$$L(y_1,\ldots,y_n,d_1,\ldots,d_n)=\begin{cases}n \text{ if for all } s\in\{1,\ldots,n\}\ y_s>d_s\\ n-1 \text{ if for exactly } n-1\ s\in\{1,\ldots,n\}\ y_s>d_s\\ \ldots\\ 2 \text{ if for exactly two } s\in\{1,\ldots,n\}\ y_s>d_s\\ 1 \text{ if for exactly one } s\in\{1,\ldots,n\}\ y_s>d_s\\ 0 \text{ if for all } s\in\{1,\ldots,n\}\ y_s\le d_s\end{cases} \tag{3}$$

If we consider all $y_s\in\tilde{Y}_{Z_s(k)}(p)$ and all $d_s\in\tilde{D}_s$ for $s=1,2,\ldots,n$, function (3) generates for a given permutation *p*, according to the Zadeh extension principle, a fuzzy set of the following form

$$\tilde{L}(p)=0/\mu_0(p)+1/\mu_1(p)+\ldots+n/\mu_n(p) \tag{4}$$

where:

$$\mu_{l\tilde{L}}(p) = \sup_{L(y_1,\ldots,y_n,d_1,\ldots,d_n)=l} \min\left(\mu_{\tilde{Y}_{Z_1(k)}(p)}(y_1),\ldots,\mu_{\tilde{Y}_{Z_n(k)}(p)}(y_n),\mu_{\tilde{D}_{s1}}(d_1),\ldots,\mu_{\tilde{D}_n}(d_n)\right)$$

The fuzzy set (4) takes on value l $(l = 0, 1, \ldots, n)$ if exactly l jobs are late.

The following is true:

$$\tilde{L}(p) = \sum_{k=1}^{n}\sum_{s=1}^{n}\tilde{L}_{Z_s(k)}(p)\cdot x_{sk}(p)$$

where:

$$\sum_{s=1}^{n} x_{sk}(p) = 1 \quad \text{and} \quad \sum_{k=1}^{n} x_{sk}(p) = 1$$

and $x_{sk}(p)$ takes on value 1 when job Z_s is processing in the k-th position for permutation p and value 0 otherwise.

As mentioned above, our optimality criterion is the expected value of $\tilde{L}(p)$, which takes on the following form:

$$E(\tilde{L}(p)) = \sum_{k=1}^{n}\sum_{s=1}^{n} E(\tilde{L}_{Z_s(k)}(p))\cdot x_{sk}(p)$$

Since

$$E(\tilde{L}_{Z_s(k)}(p)) = \frac{1}{2}\left(1 + \mu_{1_{Z_s(k)}}(p) - \mu_{0_{Z_s(k)}}(p)\right) \quad \text{for} \quad s = 1, 2, \ldots, n \tag{5}$$

we have

$$E(\tilde{L}(p)) = \frac{1}{2}\cdot\sum_{s=1}^{n} x_{sk}(p) + \frac{1}{2}\cdot\sum_{k=1}^{n}\sum_{s=1}^{n}\left[\left(\mu_{1_{Z_s(k)}}(p) - \mu_{0_{Z_s(k)}}(p)\right)\cdot x_{sk}(p)\right]$$

and, obviously,

$$E(\tilde{L}(p)) = \frac{1}{2} + \frac{1}{2}\cdot\sum_{k=1}^{n}\sum_{s=1}^{n}\left[\left(\mu_{1_{Z_s(k)}}(p) - \mu_{0_{Z_s(k)}}(p)\right)\cdot x_{sk}(p)\right]$$

4. Determination of the optimal processing schedule

The optimal processing sequence p^* is such a sequence for which the following condition is fulfilled:

$$E\left(\tilde{L}\left(p^{*}\right)\right)=\min_{p}\left[E\left(\tilde{L}(p)\right)\right]$$

Let us consider two permutations:

$$p' = \left(Z_{i_1}, Z_{i_2}, \ldots, Z_{i_{k-1}}, Z_i, Z_j, Z_{i_{k+2}}, \ldots, Z_{i_n}\right),$$

$$p'' = \left(Z_{i_1}, Z_{i_2}, \ldots, Z_{i_{k-1}}, Z_j, Z_i, Z_{i_{k+2}}, \ldots, Z_{i_n}\right).$$

The only difference between the two sequences are the k-th and k + 1 positions of Z_i and Z_j.

Theorem 1. Job Z_i should be processed before job Z_j if

$$\frac{1}{2}\cdot\left(1+\mu_{1_{Z_i(k+1)}}(p'')-\mu_{0_{Z_i(k+1)}}(p'')\right)+\frac{1}{2}\cdot\left(1+\mu_{1_{Z_j(k)}}(p'')-\mu_{0_{Z_j(k)}}(p'')\right)-$$

$$\frac{1}{2}\cdot\left(1+\mu_{1_{Z_i(k)}}(p')-\mu_{0_{Z_i(k)}}(p')\right)-\frac{1}{2}\cdot\left(1+\mu_{1_{Z_j(k+1)}}(p')-\mu_{0_{Z_j(k+1)}}(p')\right)\geq 0$$

otherwise job Z_j should be processed before job Z_i.

Proof. Sequence p' is better than p'' if

$$E\left(\tilde{L}(p'')\right)-E\left(\tilde{L}(p')\right)\geq 0$$

The proof follows immediately from (5) and the formula which comes immediately after it.

In the following we consider a special case which may occur pretty often in practice. For this special case the optimal sequence can be found by means of a well-known polynomial algorithm.

We assume that all the jobs Z_s, $s=1,2,\ldots,n$, have the same processing times $\tilde{T}(x)$. If it is so, it is easy to notice that $\tilde{L}_{Z_s(k)}(p)$ is the same for all the permutations in which job Z_s is served in the k-th position. In this case, we can introduce the notation $\tilde{L}_{Z_s(k)}$, for the common value of $\tilde{L}_{Z_s(k)}(p)$ for all the permutations:

$$(Z_{i_1}, Z_{i_2}, \ldots, Z_{i_{k-1}}, Z_s, Z_{i_{k+2}}, \ldots, Z_{i_n})$$

The following is true:

$$\tilde{L}_{Z_s(k)} = 0/\mu_{0_{Z_s(k)}} + 1/\mu_{1_{Z_s(k)}}$$

Theorem 2. If all the jobs Z_s, $s=1,2,\ldots,n$.., have the same processing times $\tilde{T}(x)$, then the optimal sequence according to the criterion of the expected value of the number of late jobs can be found by means of a minimal assignment algorithm (e.g., the Hungarian method), where jobs $Z_1, Z_2, \ldots, Z_n$ are assigned to their position in the optimal sequence and the elements c_{sk} of the assignment matrix will be equal to

$$E\left(\tilde{L}_{Z_s(k)}\right).$$

Proof. For each permutation we have:

$$\min_p E\left(\tilde{L}(p)\right) = \min_p \sum_{k=1}^{n}\sum_{s=1}^{n} E\left(\tilde{L}_{Z_s(k)}\right)\cdot x_{sk}(p) = \min_{x_{sk}} \sum_{k=1}^{n}\sum_{s=1}^{n}\left[E\left(\tilde{L}_{Z_s(k)}\right)\right]\cdot x_{sk}$$

where:

$$\sum_{s=1}^{n} x_{sk} = 1 \quad \text{and} \quad \sum_{k=1}^{n} x_{sk} = 1,$$

which completes the proof.

Let us now come back to the general case.

5. Determining the fuzzy set of the number of late jobs

In this section, we will show how fuzzy set (4), representing the number of late jobs for a given permutation p, can be determined by means of a simple procedure.

In order to determine for a given permutation p the membership function of the fuzzy set of number of late jobs $\tilde{L}(p)$, it is enough to follow the following procedure:

1) For each job Z_s $(s = 1, \ldots, n)$ determine, according to the job position in the sequence, fuzzy set $\tilde{L}_{Z_s(k)} = 0/\mu_{0_{Z_s(k)}}(p) + 1/\mu_{1_{Z_s(k)}}(p)$ (on the basis of (2)).

2) Order values $\mu_{1_{Z_s}}(p)$ in a non-increasing way, adding element equal to 1 at the beginning of the sequence. In this way, the sequence $\Lambda_0(p), \Lambda_1(p), \ldots, \Lambda_n(p)$ is formed.

3) Order values $\mu_{0_{Z_s}}(p)$ in a non-decreasing way, adding element equal to 1 at the end of the sequence. In this way, the sequence $\lambda_0(p), \lambda_1(p), \ldots, \lambda_n(p)$ is formed.

4) Determine values of the membership function of fuzzy set $\tilde{L}(p)$, which determines the possibility of exactly l jobs being late for permutation p, according to the formula: $\mu_l(p) = \min\left(\Lambda_l(p), \lambda_l(p)\right)$ for $l = 0, 1, \ldots, n$.

Remarks. In the sequence $\Lambda_0(p), \Lambda_1(p), \ldots, \Lambda_n(p)$ l-th element gives the possibility of at least l jobs being late for permutation p.

In the sequence $\lambda_0(p), \lambda_1(p), \ldots, \lambda_n(p)$ is formed, whose *l*-th element gives the possibility of at least *l* jobs being finished on time for permutation *p*.

The correctness of the algorithm can be proved by mathematical induction.

6. Numerical example

Let us consider jobs Z_1, Z_2, Z_3. For each job Z_s its processing time $\tilde{T}_s$ is a triangular fuzzy number $\mu_{\tilde{T}_s}(x)$ = (1, 3, 2). The due dates are crisp and take on, respectively, the following values $\tilde{D}_1 = (4, 4, 4)$, $\tilde{D}_2 = (6, 6, 6)$, $\tilde{D}_3 = (8,8,8)$. All the jobs are ready to be processed at the moment 0.

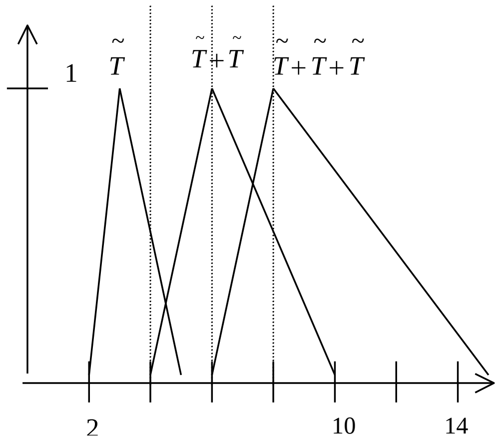

Fig. 1. Illustration of the example

The expected values of the jobs being late, for each possible job position, are presented in the following table (they result from formula (5)):

Z_s/k	1	2	3
Z_1	0,25	1	1
Z_2	0	0,5	1
Z_3	0	0,25	0,5

The sequences minimizing the expected number of late jobs are the following ones: $p^* = (Z_1, Z_2, Z_3)$ and $p^* = (Z_2, Z_3, Z_1)$. The expected number of late jobs for the optimal sequences is equal to 1.25.

For the optimal permutation $p^* = (Z_2, Z_3, Z_1)$, the fuzzy sets giving for each individual job the possibility of the job being late are as follows:

$$\widetilde{L}_{Z_2(1)}(p^*) = 0/1 + 1/0$$

$$\widetilde{L}_{Z_3(2)}(p^*) = 0/1 + 1/0.5$$

$$\widetilde{L}_{Z_1 3}(p^*) = 0/0 + 1/1.$$

And the fuzzy set of the number of late jobs for this permutation is

$$\widetilde{L}(p^*) = 0/0 + 1/1 + 2/0.5 + 3/0\,.$$

The procedure proposed informs the decision maker how many jobs can be late for a given permutation. This information can also be found for the optimal permutation according to any criteria chosen by the decision maker (here, we use the criteria of the expected value of the fuzzy number of jobs being late). It allows us to assess the risk of "too many" jobs being late, where the meaning of the expression "too many" will of course be understood as a function of the given decision situation.

Conclusions

We have considered one-machine scheduling problem with fuzzy processing times and fuzzy due dates. We have shown how to determine the optimal sequence according to the criterion of the minimal expected number of late jobs. For a special case we have shown how the optimal schedule can be determined effectively by means of a well-known algorithm. We have also shown how to calculate the membership function of the fuzzy set of the number of late jobs for a given permutation.

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Minimalizacja oczekiwanej liczby opóźnionych zadań w systemach obsługi z jedną maszyną z rozmytymi czasami obsługi i terminami dyrektywnymi

W artykule rozważany jest problem obsługi zadań w systemach obsługi z jedną maszyną. Przyjmuje się, że czasy obsługi zadań oraz terminy dyrektywne są liczbami rozmytymi. Jako kryterium optymalizacji przyjęto minimalną średnią liczbę opóźnionych zadań. W konstrukcji rozwiązania zastosowano zasadę rozszerzenia Zadeha. Rozważa się również szczególne przypadki problemu, które można zredukować do znanych łatwych problemów optymalizacyjnych. Zaproponowano tu zastosowanie klasycznego modelu przydziału zadań.

W pracy zaprezentowano także algorytm wyznaczania postaci rozmytego zbioru liczby opóźnionych zadań. Bazuje on na mocy rozmytych zbiorów liczby zadań opóźnionych oraz liczby zadań wykonanych w terminie.